*Journal of Statistical Physics, Vol. 80, Nos. 5/6, 1995*

# Metastable Behavior of Low-Temperature Glauber Dynamics with Stirring

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*Received May 19, 1994; final March 17, 1995*

We consider the metastable behavior of a superposition of a ferromagnetic spin system with a Glauber dynamics and stirring dynamics. Starting from configuration $-\underline{1}$, minus spins at all lattice sites in a fixed volume under periodic boundary conditions, the process stays close to this configuration for an unpredictable time until the formation of a droplet, of spins $+1$, with a certain critical size and decays to configuration $+\underline{1}$ in a relatively short time. We observe that the size of the droplet depends on the rate of exclusion.

**KEY WORDS:** Ferromagnetic spin system; stirring dynamics; metastability.

## 0. INTRODUCTION

The two-dimensional nearest neighbor ferromagnetic Ising model, with an external magnetic field, evolving according to a Glauber dynamics (i.e., a reversible spin-flip dynamics) was studied in ref. 6; it presents a metastable behavior.

Here, we deal with the model obtained by superposing onto this a symmetric simple exclusion process. Our motivation is to determine, in the limit as the temperature goes to zero, for fixed finite volume, if this model still presents the essential features that one associates with metastability. We choose the rate of symmetric simple exclusion competing with the corner erosion of ref. 6.

In fact, if the external field is small and positive ($0 < h < 1$), the system, when started from the configuration with all spins down, behaves as if it were in a steady state for a very long time until a droplet is formed. Then,

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in a relatively short time, it evolves to the configuration with all spins up. In the present case, the size of the above droplet does not depend on $h$.

Moreover, our model is not reversible and we do not know explicitly its unique invariant measure, in contrast to ref. 6, where this was indeed used.

The characterization of metastable behavior in the above fashion was introduced in ref. 1, to which we refer for more general discussions.

In ref. 8 the reader will find a procedure of renormalization of finite state-space Markov chains with transition probabilities exponentially small in a parameter $\bar{\beta}$; the main results in ref. 6 were rederived using such a method. This procedure gives us an estimate of the time needed for our process to reach the configuration with all spins up $(+\underline{1})$ when starting from the configuration with all spins down $(-\underline{1})$.

To obtain the stability of time averages we need basically two things: an upper bound on the time of *thermalization* (here to go back to $-\underline{1}$) and a lower bound on the time of *tunneling*. For this last one we will use the above estimate.

As in our model there are equivalence classes, we cannot use Theorem 2.2 in ref. 8 to prove the asymptotic exponential behavior of the time until it reaches $+\underline{1}$ starting from $-\underline{1}$, normalized by its mean. Thus we present another proof of this result.

This paper is organized in the following way. We introduce the model and the dynamics and summarize our results in Section 1. In Section 2 we comment on the decomposition of the set of configurations and in Section 3 we prove the results. In the last section we discuss smaller rates of the symmetric simple exclusion process, where the exact picture of ref. 6 is preserved.

## 1. MODEL AND RESULTS

We consider the two-dimensional nearest neighbor ferromagnetic Ising model on a finite torus $\Lambda_N$, with a random perturbation given by a symmetric simple exclusion process.

The process takes values on $X_N = \{-1, +1\}^{\Lambda_N}$, where $\Lambda_N = \{1,..., N\}^2$ and its generator acts on functions $f$ as

$$Lf(\eta) = \sum_{x \in \Lambda_N} c(x, \eta)[f(\eta^x) - f(\eta)] + \sum_{x, y \in \Lambda_N} c(x, y, \eta)[f(\eta^{x,y}) - f(\eta)] \tag{1.1}$$

where $c(x, \eta)$ and $c(x, y, \eta)$ are the rates associated to Glauber dynamics and to stirring dynamics, respectively. We will only consider the Glauber dynamics for which the spin at site $x$, when the configuration is $\eta \in X_N$, flips at rate

$$c(x, \eta) = \begin{cases} 1 & \text{if} \quad \Delta_x H(\eta) \leqslant 0 \\ \exp\{-\beta \Delta_x H(\eta)\} & \text{otherwise} \end{cases} \tag{1.2}$$

where $H(\eta)$, the Hamiltonian at configuration $\eta$, is given by

$$H(\eta) = -\frac{1}{2} \sum_{\langle x, y \rangle} \eta(x)\, \eta(y) - \frac{h}{2} \sum_{x \in \Lambda_N} \eta(x) \tag{1.3}$$

The first sum in the Hamiltonian runs over the pairs of nearest neighbor sites of $\Lambda_N$, counting each pair only once, and

$$\Delta_x H(\eta) = H(\eta^x) - H(\eta)$$

with

$$\eta^x(y) = \begin{cases} \eta(y) & \text{if} \quad x \neq y \\ -\eta(y) & \text{otherwise} \end{cases}$$

We consider $\beta > 0$ the inverse temperature.

For the stirring dynamics we will associate the rate

$$c(x, y, \eta) = \begin{cases} \exp\{-\beta h\} & \text{if} \quad \|x - y\| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

with

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if} \quad z \neq x, z \neq y \\ \eta(x) & \text{if} \quad z = y \\ \eta(y) & \text{if} \quad z = x \end{cases}$$

and $\|x\| = |x_1| + |x_2|$.

For each initial configuration $\eta$, these rates define a continuous-time Markov process $(\sigma_t^\eta,\ t \geqslant 0)$ such that at $t = 0$, $\sigma_t^\eta = \eta$ with probability one and for $\xi \neq \zeta$ and $\varepsilon > 0$

$$\mathbb{P}(\sigma_{t+\varepsilon}^\eta = \xi \mid \sigma_t^\eta = \zeta) = \begin{cases} c(x, \zeta)\, \varepsilon + o(\varepsilon) & \text{if} \quad \xi = \zeta^x \text{ for some } x \in \Lambda_N \\ c(x, y, \zeta)\, \varepsilon + o(\varepsilon) & \text{if} \quad \xi = \zeta^{x,y},\ \|x - y\| = 1 \\ o(\varepsilon) & \text{otherwise} \end{cases} \tag{1.5}$$

For each $\eta \in X_N$ and $A \subset X_N$, we define the hitting time

$$T^\eta(A) = \inf\{t \geqslant 0 \colon \sigma_t^\eta \in A\} \tag{1.6}$$

Our main goal is to describe the behavior of the system when it starts from $-\underline{1}$ until it reaches $+\underline{1}$ for $N$ and $h$ both fixed and $\beta$ going to infinity.

If $h > 4$ any spin $-1$ will flip with rate 1 even if its neighbors are also $-1$, and any spin $+1$ will flip with a vanishing rate as $\beta \to \infty$. In this case the action of the stirring process is not seen before $T^{-\underline{1}}(+\underline{1})$ because

$$\lim_{\beta \to \infty} \mathbb{P}(T^{-\underline{1}}(+\underline{1}) > \exp\{\beta(h-\delta)\}) = 0, \qquad \forall 0 < \delta < h$$

In fact $T^{-\underline{1}}(+\underline{1})$ in this case is polynomial in $N$ independent of $\beta$.

If $2 < h < 4$ and $N \geqslant 3$ after reaching the state with only one spin $+1$, four things can happen. With rate 1 the system goes back to $-\underline{1}$, with rate 4 it goes to a configuration with two neighbor $+1$ spins, with rate $(N^2 - 5)\exp\{-\beta(4-h)\}$ it goes to the other configuration with two spins $+1$, and with rate $5\exp\{-\beta h\}$ the exclusion process occurs involving the unique $+1$ spin and basically nothing happens.

As the third and fourth possibilities have vanishing rate (as $\beta \to \infty$), only the first two are important. If the system goes back to $-\underline{1}$, everything starts anew, but if it gets to a configuration with two neighbor $+1$ spins, then both of them flip back with vanishing rate, too $[\exp\{-\beta(h-2)\}]$, while its neighbors flip from $-1$ to $+1$ with rates 1. So the pair of neighbor spins $+1$ forms a critical droplet that nucleates the passage from $-\underline{1}$ to $+\underline{1}$.

We now investigate the general picture of the passage from $-\underline{1}$ to $+\underline{1}$ for smaller magnetic field, i.e., $0 < h < 1$.

From ref. 6 we know the behavior of the process given by $L = L_G$, where $L_G$ is the generator associated to Glauber dynamics defined above. The question here is: *What happens to this behavior when we add the stirring perturbation?* Now, the size of the critical droplet does not depend on $h$.

Before stating Theorem 1.1, we need some definitions. Let $\mathscr{R}$ be the set of configurations with all spins $-1$ except for those in a rectangle $l_1 \times l_2$, which are $+1$, with $l_1$ and $l_2$ less than $N-1$. For $\eta \in \mathscr{R}$ define $l(\eta) = \min(l_1, l_2)$.

**Theorem 1.1.** Suppose that $0 < h < 1$ and $\eta \in \mathscr{R}$.

(a) If $l(\eta) > 3$, then

$$\lim_{\beta \to \infty} \inf \mathbb{P}(T^\eta(+\underline{1}) < T^\eta(-\underline{1})) = 1 \tag{1.7}$$

(b) If $l(\eta)=1$, then

$$\lim_{\beta\to\infty} \inf \mathbb{P}(T^{\eta}(-\underline{1})<T^{\eta}(+\underline{1}))=1 \tag{1.8}$$

(c) If $l(\eta)=2$ or $l(\eta)=3$, then

$$\begin{aligned}&\lim_{\beta\to\infty} \inf \mathbb{P}(T^{\eta}(+\underline{1})<T^{\eta}(-\underline{1}))>0\\&\lim_{\beta\to\infty} \inf \mathbb{P}(T^{\eta}(-\underline{1})<T^{\eta}(+\underline{1}))>0\end{aligned} \tag{1.9}$$

Theorem 1.1 has as a consequence the possibility of dividing the set of all configurations into three nonempty sets $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ such that $\mathscr{A}$ and $\mathscr{C}$ are, respectively, the basins of attraction of $-\underline{1}$ and $+\underline{1}$, while starting from $\mathscr{B}$, the system can go to $\mathscr{A}$ or $\mathscr{C}$, both with comparable probabilities as $\beta$ increases.

**Proposition 1.1.** Suppose $0<h<1$. The set of configurations can be partitioned into three nonempty sets $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ such that:

(a) If $\eta\in\mathscr{A}$ and $\varepsilon>0$, then

$$\lim_{\beta\to\infty} \mathbb{P}(T^{\eta}(-\underline{1})<T^{\eta}(+\underline{1}),\ T^{\eta}(-\underline{1})<e^{\beta\varepsilon})=1 \tag{1.10}$$

(b) If $\eta\in\mathscr{C}$ and $\varepsilon>0$, then

$$\lim_{\beta\to\infty} \mathbb{P}(T^{\eta}(+\underline{1})<T^{\eta}(-\underline{1}),\ T^{\eta}(+\underline{1})<e^{\beta(h+\varepsilon)})=1 \tag{1.11}$$

(c) If $\eta\in\mathscr{B}$ and $\varepsilon>0$, then

$$\lim_{\beta\to\infty} \inf \mathbb{P}(T^{\eta}(-\underline{1})<T^{\eta}(+\underline{1}))>0$$

$$\lim_{\beta\to\infty} \inf \mathbb{P}(T^{\eta}(+\underline{1})<T^{\eta}(-\underline{1}))>0$$

and for every $\varepsilon>0$

$$\lim_{\beta\to\infty} \mathbb{P}(T^{\eta}(\{-\underline{1},+\underline{1}\})<e^{\beta(h+\varepsilon)})=1 \tag{1.12}$$

The next two theorems characterize the metastable behavior of $(\sigma_t^{-\underline{1}})_{t\geqslant 0}$.

**Theorem 1.2.** Consider $T = \inf\{t \geqslant 0: \sigma_t^{-1} = +\underline{1}\}$; then

$$\frac{T}{E(T)} \to \tau \quad \text{in distribution as} \quad \beta \to \infty \tag{1.13}$$

where $\tau$ is a unit mean exponential random variable.

Before stating Theorem 1.3 (stability of time averages), we introduce the following notation:

- $T^\eta = T^\eta(+\underline{1})$.
- $\tilde{T}^\eta = T^\eta(-\underline{1})$.
- $\mathscr{M}_1$ denotes the space of probability measures on $X_N$.
- $C_b(X_N)$ denotes the space of bounded (continuous) real functions on $X_N$.

**Theorem 1.3.** Let $\gamma_\beta$ be defined through the relation $\mathbb{P}(T > \gamma_\beta) = e^{-1}$. It is possible to find $R_\beta > 0$ with $R_\beta \to \infty$ and $R_\beta/\gamma_\beta \to 0$ as $\beta \to \infty$, so that if we define the $\mathscr{M}_1$-valued processes $(\nu_t^\beta)_{t \geqslant 0}$ via

$$\nu_t^\beta(f) = \frac{1}{R_\beta} \int_{t\gamma_\beta}^{t\gamma_\beta + R_\beta} f(\sigma_u)\, du, \qquad f \in C_b(X_N) \tag{1.14}$$

then, for each initial configuration $\eta \in \mathscr{A}$,

$$\mathbb{P}\{\sup_{0 \leqslant s \leqslant (T^\eta - 3R_\beta)/\gamma_\beta} |\nu_s^\beta(f) - f(-\underline{1})| > \delta\} \to 0 \tag{1.15}$$

$$\mathbb{P}\{\sup_{T^\eta/\gamma_\beta \leqslant s \leqslant (\tilde{T}^\eta - 3R_\beta)/\gamma_\beta} |\nu_s^\beta(f) - f(+\underline{1})| > \delta\} \to 0 \tag{1.16}$$

as $\beta \to \infty$ for each $f \in C_b(X_N)$ and each $\delta > 0$. Finally, let

$$\begin{aligned} \tilde{\nu}_t^\beta &= \nu_t^\beta \qquad \text{if} \quad t \notin \left[\frac{T^\eta - 3R_\beta}{\gamma_\beta}, \frac{T^\eta}{\gamma_\beta}\right) \\ &= \nu^\beta_{(T^\eta - 3R_\beta)/\gamma_\beta} \qquad \text{otherwise} \end{aligned} \tag{1.17}$$

Then, for each $\eta \in \mathscr{A}$, $(\tilde{\nu}_t^\beta)_{t \geqslant 0}$ converges in law on $D([0, +\infty), \mathscr{M}_1)$ to a Markov jump process $(\nu_t)_{t \geqslant 0}$ given by

$$\begin{aligned} \nu_t &= \delta_{-\underline{1}} \qquad \text{if} \quad t < \tau \\ &= \delta_{+\underline{1}} \qquad \text{if} \quad t > \tau \end{aligned} \tag{1.18}$$

where $\tau$ is an exponential random variable with mean one.

## 2. CHARACTERIZATION OF THE SETS $\mathscr{A}$, $\mathscr{B}$, AND $\mathscr{C}$

For $\eta \in \{-1, +1\}^{\Lambda_N}$ define $T_+\eta$ as the configuration obtained from $\eta$ by flipping all the spins $-1$ with at least two opposite neighbors. Define also $T_-\eta$ as the configuration obtained from $\eta$ by flipping all the spins $+1$ with at least three opposite neighbors. Applying iteratively $T_+$ (respectively $T_-$) on $\eta$, we obtain an increasing (decreasing) sequence of configurations that becomes constant after a finite number of applications. Denote by $\bar{\eta}$ (resp. $\underline{\eta}$) this final configuration.

If $0 < h < 1$, each one of the flips defining $T_+$ and $T_-$ occurs with rate 1 for our Glauber dynamics. So one has the following result.

**Lemma 2.1.** If $0 < h < 1$ and $0 < \delta < h$, then

$$\begin{aligned} \inf_{\beta \geq 0} \inf_{\eta \in X_N} \mathbb{P}(A) &:= \alpha_+ > 0 \\ \inf_{\beta \geq 0} \inf_{\eta \in X_N} \mathbb{P}(B) &:= \alpha_- > 0 \end{aligned} \tag{2.1}$$

where $A$ and $B$ are the following events:

$$A := \{\sigma_s^\eta = \bar{\eta} \text{ for some } s \in [0, e^{\beta\delta}];\ \sigma_{e^{\beta\delta}}^\eta = \underline{(\bar{\eta})}\}$$

$$B := \{\sigma_s^\eta = \underline{\eta} \text{ for some } s \in [0, e^{\beta\delta}];\ \sigma_{e^{\beta\delta}}^\eta = \overline{(\underline{\eta})}\}$$

*Proof.* It is easy to see that $\alpha_+$ is the probability of all flips such that $\eta \to \underline{(\bar{\eta})}$, where each step occurs with rate 1. ∎

By Lemma 2.1, in a time of order $e^{\beta h}$, for any $\delta < h$, the system started from $\eta \in \{-1, +1\}^{\Lambda_N}$ can go with nonvanishing probability, as high as $\bar{\eta}$, but not higher. So $\mathscr{A}$ must be the set of configurations $\eta$ such that starting from $\bar{\eta}$, we are still likely to go to $-\underline{1}$ before $+\underline{1}$.

So we define $\mathscr{A}$ as the set of configurations $\eta$ such that $\underline{(\bar{\eta})}$ is equal to the configuration $-\underline{1}$.

Again by Lemma 2.1, the system starting at $\eta$ can go with nonvanishing probability in a time of order $e^{\beta h}$, for any $\delta < h$, as low as $\underline{\eta}$, but not lower. But even if $\underline{\eta}$ is reached the system is likely to go to $\overline{(\underline{\eta})}$ in a time of order $e^{\beta h}$, for any $\delta < h$. We define $\mathscr{C}$ as the set of configurations $\eta$ such that at least one of the droplets of $+1$ spins in $\overline{(\underline{\eta})}$ is a rectangle with all sides larger than 3 or a ring of width larger than 1 around the torus.

Finally, $\mathscr{B}$ is the set of configurations not in $\mathscr{A}$ or in $\mathscr{C}$. Starting from $\eta \in \mathscr{B}$, we can reach either $-\underline{1}$ or $+\underline{1}$.

## 3. PROOF OF RESULTS

### 3.1. Proof of Theorem 1.1

If $\eta \in \mathscr{R}$, we may suppose without loss of generality that we have $l_1$ and $l_2$ with $l(\eta) = l_1 \leqslant l_2$ such that all the spins in $\eta$ are $-1$ except those inside the rectangle $R = \{1,..., l_1\} \times \{1,..., l_2\}$. We also define the slices of $R$ to be the sets $H_j = \{1,..., l_1\} \times \{j\}$ and $V_i = \{i\} \times \{1,..., l_2\}$ with $j \in \{1,..., l_2\}$ and $i \in \{1,..., l_1\}$.

First, note that $Y_t = \#\{x \in H_1: \sigma_t(x) = -1\}$ is not a Markov process, because its jump rates depend on configuration. Moreover, a jump followed by the action of $T_-$ can destroy at most three positive spins when these form an isolated block. In this manner we obtain

$$Y_t \leqslant \tilde{Y}_t \overset{\text{def}}{=} \#\{x \in H_1: (T_-\sigma_t)(x) = -1 \text{ or } \sigma_t(x) = -1\}$$

To prove part (a) of the theorem, we need to compare the number of negative spins in $H_1$ with the process $\{X_t\}_{t \geqslant 0}$, where $X_0 = 0$ and its jump rates are

$$c_X(n, n+3) = 3le^{-\beta h} + (l-2)\, e^{-\beta(2+h)}$$
$$c_X(n, n-1) = 1$$

Note that

$$c_X(n, n+3) \geqslant \max_\eta c_{\tilde{Y}}(n, n+1, \eta) + \max_\eta c_{\tilde{Y}}(n, n+2, \eta) + \max_\eta c_{\tilde{Y}}(n, n+3, \eta)$$

and (3.1.1)

$$c_X(n, n-1) \leqslant \min_\eta c_{\tilde{Y}}(n, n-1, \eta)$$

Now, define

$$\Theta_Y = \inf\{t \geqslant 0: |Y_t| = l-1\}$$
$$\Theta_X = \inf\{t \geqslant 0: X_t \geqslant l-1\}$$

As a consequence of (3.1.1) we have that

$$\mathbb{P}(\Theta_Y > t) \geqslant \mathbb{P}(\Theta_X > t)$$

To estimate $\lim_{\beta\to\infty} \mathbb{P}(\Theta_X > t)$, consider $T_1, T_2,...,$ the times when some modification occurs in $X_t$. We know that $T_1$, $T_k - T_{k-1}$, $k = 2, 3,...,$ are i.i.d. random variables with exponential distribution of rate $[1 + c_X(n, n+3)]$; we have

$$P(X_{T_k} = X_{T_{k-1}} - 1) = \frac{1}{1 + c_X(n, n+3)} = q$$

$$P(X_{T_k} = X_{T_{k-1}} + 3) = \frac{c_X(n, n+3)}{1 + c_X(n, n+3)} = p$$

Now, for any $M > \lceil l/3 \rceil + 1$,

$$\begin{aligned}
\mathbb{P}(\Theta_X < t) &= \sum_N \mathbb{P}(\Theta_X < t,\ T_N \leqslant t < T_{N+1}) \\
&\leqslant \sum_{N=\lceil l/3\rceil+1} \mathbb{P}(\Theta_X < t \mid T_N \leqslant t < T_{N+1})\, \mathbb{P}(T_N \leqslant t < T_{N+1}) \\
&\leqslant \sum_{N=\lceil l/3\rceil+1}^{M} \mathbb{P}(\Theta_X < t \mid T_N \leqslant t < T_{N+1}) + \sum_{N>M} \mathbb{P}(T_N \leqslant t < T_{N+1}) \\
&\leqslant \sum_{N=\lceil l/3\rceil+1}^{M} \mathbb{P}\left(X_{T_i} \geqslant l - 1 \text{ for some } i \in \left\{\left\lceil \frac{l}{3} \right\rceil + 1,..., N\right\}\right) \\
&\quad + \sum_{N>M} \mathbb{P}(T_N \leqslant t < T_{N+1}) \\
&\leqslant p^{\lceil l/3\rceil+1} + Mp^{\lceil l/3\rceil+1} + 2Mp^{\lceil l/3\rceil+1} \\
&\quad + \sum_{N=\lceil l/3\rceil+4}^{M} Np^{\lceil l/3\rceil+1} p^{\lceil (N-\lceil l/3\rceil+1)/3\rceil} \\
&\quad + \sum_{N>M} \mathbb{P}(T_N \leqslant t < T_{N+1}) \\
&\leqslant 3Mp^{\lceil l/3\rceil+1} + Mp^{\lceil l/3\rceil+1} \sum_{N=1}^{M} 3p^N + \sum_{N>M} \mathbb{P}(T_N \leqslant t < T_{N+1})
\end{aligned}$$

Taking $t = \exp\{\beta[h(\lceil l/3\rceil + 1) - \varepsilon]\}$ and $M = \lceil [1 + c_X(n, n+3) + \delta]\, t \rceil$, we have that this upper bound tends to zero when $\beta$ tends to infinity, for any $\delta > 0$.

Moreover, given $\delta_1 > 0$, a protuberance occurs in a time of order $l \exp\{\beta(h + \delta_1)\}$, with probability close to one; this implies that a new slice will be created before $t = \exp\{\beta[h(\lceil l/3\rceil + 1) - \varepsilon]\}$.

In this manner the process reaches a configuration with a larger rectangular droplet and by the strong Markov property the process restarts from this.

For part (b) of Theorem 1.1 it is enough to verify that the erosion occurs in a time of order $e^{\beta\delta}$, $0<\delta<h$, i.e., a time smaller than $e^{\beta h}$ with probability close to one when $\beta$ tends to infinity and, since the jump and creation rates are smaller than $e^{-\beta\delta}$, $0<\delta<h$, we have the result.

For part (c) it is enough to verify that a jump following the action of $T_-$ can destroy a slice of size 3 and a jump following the action of $T_+$ can create a new slice. The result follows from Lemma 2.1: ∎

## 3.2. Proof of Proposition 1.1

(a) Using Theorem 1.1 and the fact that $\underline{\eta}$ is reached in a time of order 1, we have the result.

(b) Using Theorem 1.1 and the fact that a new slice is created in a time of order $e^{\beta(h+\delta)}$, $\forall\delta>0$, we have the result.

Before proving part (c) of Proposition 1.1 we introduce a definition.

**Definition 3.2.1.** A configuration $\eta$ is said to be stable if any spin $+1$ has at least two positive neighbors and any spin $-1$ has at least three negative neighbor spins. Thus, a configuration $\eta$ is stable if $\eta=\bigcup_i R^i$, where $R^i$ are rectangles on the lattice of sides $l_1^i \leqslant l_2^i$ with $l_1^i>1$ for any $i$, and $\operatorname{dist}(R^i, R^j)>2$ if $i\neq j$.

Considering $\eta\in\mathscr{B}$, then $\underline{(\bar{\eta})}$ and $\overline{(\underline{\eta})}$ are stable with $2\leqslant l_1^k\leqslant 3$, $\forall k$.
Now, define the stopping time

$$T_{\mathscr{S}}=\inf\{t\geqslant 0: \sigma_t^\eta\in\mathscr{S}\} \tag{3.2.1}$$

We shall prove that

$$\lim_{\beta\to\infty} \mathbb{P}(T_{\mathscr{A}\cup\mathscr{C}}<e^{\beta(h+\varepsilon)})=1 \tag{3.2.2}$$

As any configuration in $\mathscr{B}$, by Lemma 2.1, reaches a stable configuration in $\mathscr{B}$ in a time of order 1, we will compare our process with the Markov chain whose state space is the set of stable configurations in $\mathscr{B}$, $\mathscr{A}$, and $\mathscr{C}$, as follows:

$$\begin{array}{cccccc} \mathscr{A} & \eta_{2\times 2} & \eta_{2\times 3} & \cdots & \eta_{2\times(N-2)} & \mathscr{C} \\ & \eta_{3\times 3} & \eta_{3\times 4} & \cdots & \eta_{3\times(N-2)} & \end{array}$$

In fact, consider

$$t_1=\inf\{t\geqslant 0: \sigma_t^{-1} \text{ is stable}\}$$
$$t_k=\inf\{t\geqslant t_{k-1}: \sigma_t^{-1} \text{ is stable}\} \qquad \text{for} \quad k=2,\ldots$$

Then $\{\sigma_{t_i}^{-1}\}_{i\geq 1}$ is the Markov chain we want to describe (up to its first exit from $\mathscr{B}$).

Consider:

- $\mathscr{E} = \{\mathscr{A}, \mathscr{C}, \eta_{2\times 2}, \eta_{2\times 3},..., \eta_{2\times (N-2)}, \eta_{3\times 3}, \eta_{3\times 4},..., \eta_{3\times (N-2)}\}$ the space of states.
- $\mathscr{A}$ and $\mathscr{C}$ are absorbing states.
- $p(\eta, \zeta) > 0$ if $\eta$ differs from $\zeta$ by only one slice (horizontal or verical) in the droplet of positive spins or $\eta = \zeta$.

In the Markov chain described above each step waits an exponential time with mean of order $e^{\beta h}$.

Define $N(t)$ the number of steps on an interval $[0, t]$; $N(t)$ can be superestimated by a Poisson process with rate $\lambda t$, where $\lambda$ is of order $e^{-\beta h}$.

As the number of steps in $[0, e^{\beta(h+\varepsilon)}]$, $\varepsilon > 0$, goes to infinity ($\approx e^{\beta\varepsilon}$) and $\mathscr{B}$ consists of transient states, then at this time the process goes out of $\mathscr{B}$ with probability one.

To prove that

$$\lim_{\beta\to\infty} \inf \mathbb{P}(T_{\mathscr{A}} < T_{\mathscr{C}}) > 0$$

$$\lim_{\beta\to\infty} \inf \mathbb{P}(T_{\mathscr{C}} < T_{\mathscr{A}}) > 0$$

it is enough to show that each finite sequence $s = (s_1,..., s_k)$, where $s_1 = \eta$, $\eta \in \mathscr{B}$, $s_k \in \mathscr{C} \cup \mathscr{A}$, and $p(s_i, s_{i+1}) > 0$, has positive probability.

But $\mathbb{P}(s) \geq (1/N^2)^k$ $\forall k \in \mathbb{N}$, so that we have the result. ▮

### 3.3. Proof of Theorem 1.2

**Notation.** For simplicity $T^{-1}(+\underline{1}) = T$.

Set $\mathscr{D} = \mathscr{A} \cup \mathscr{B}$ and define

$$S = T - T(\mathscr{C})$$

We will prove that

$$\frac{T(\mathscr{C})}{\gamma_\beta} \to \tau \quad \text{in distribution} \tag{3.3.1}$$

where $\tau$ is a unit mean exponential random variable and $\gamma_\beta$ is defined by

$$\mathbb{P}(T(\mathscr{C}) > \gamma_\beta) = e^{-1} \tag{3.3.2}$$

Then we will prove

$$\frac{S}{\gamma_\beta} \to 0 \quad \text{in probability as} \quad \beta \to \infty \tag{3.3.3}$$

from which it follows that

$$\frac{T}{\gamma_\beta} \to \tau \quad \text{in distribution as} \quad \beta \to \infty \tag{3.3.4}$$

After doing this we need only to replace $\gamma_\beta$ by $ET$, and this will be done using a standard argument.

To prove (3.3.1), we introduce a dynamics restricted to the set $\mathscr{D}$. We present the idea in a general fashion.

We say that a set $\varphi$ of configurations is connected if for any pair of configurations $\eta_1, \eta_2 \in \varphi$ it is possible to go from $\eta_1$ to $\eta_2$ by a chain of transformations in which a single dynamics occurs at each step without leaving $\varphi$.

The Glauber + stirring dynamics restricted to $\varphi$ is defined by the rates

$$\begin{aligned} \tilde{c}(x, \eta) &= \begin{cases} c(x, \eta) & \text{if} \quad \eta, \eta^x \in \varphi \\ 0 & \text{otherwise} \end{cases} \\ \tilde{c}(x, y, \eta) &= \begin{cases} c(x, y, \eta) & \text{if} \quad \eta, \eta^{x,y} \in \varphi \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.3.5}$$

The process $(\tilde{\sigma}_t)_{t \geqslant 0}$ can be coupled with $(\sigma_t^\eta)_{t \geqslant 0}$ in a very intuitive and useful way, as in ref. 6: the two processes jump together until the latter escapes from $\varphi$; at this moment the former process stays still and afterward they evolve independently. We will use this coupling several times, so we call it coupling A, as in ref. 6.

In our case, $\varphi = \mathscr{D}$. Consider $\tilde{\mu}(\cdot)$ the invariant measure for restricted dynamics.

**Lemma 3.3.1.** Consider $T_{\mathscr{A}}^{\tilde{\mu}} = \inf\{t \geqslant 0 \colon \tilde{\sigma}_t^{\tilde{\mu}} \in \mathscr{A}\}$; then

$$\lim_{\beta \to \infty} \mathbb{P}(T_{\mathscr{A}}^{\tilde{\mu}} < e^{\beta(h+\delta)}) = 1, \quad \forall \delta > 0$$

*Proof.*

$$\begin{aligned} \mathbb{P}(T_{\mathscr{A}}^{\tilde{\mu}} < e^{\beta(h+\delta)}) &= \mathbb{P}(T_{\mathscr{A}}^{\tilde{\mu}} < e^{\beta(h+\delta)}, \eta_0 \in \mathscr{A}) + \mathbb{P}(T_{\mathscr{A}}^{\tilde{\mu}} < e^{\beta(h+\delta)}, \eta_0 \in \mathscr{B}) \\ &= \tilde{\mu}(\mathscr{A}) + \sum_{\sigma \in \mathscr{B}} \mathbb{P}(T_{\mathscr{A}}^{\sigma} < e^{\beta(h+\delta)})\, \mathbb{P}(\eta_0 = \sigma) \end{aligned}$$

By Proposition 1.1,

$$\lim_{\beta \to \infty} \mathbb{P}(T^{\sigma}_{\mathscr{A}} < e^{\beta(h+\delta)}) = 1 \qquad \text{if} \quad \sigma \in \mathscr{B}$$

Then,

$$\lim_{\beta \to \infty} \mathbb{P}(T^{\tilde{\mu}}_{\mathscr{A}} < e^{\beta(h+\delta)}) = \lim_{\beta \to \infty} [\tilde{\mu}(\mathscr{A}) + \tilde{\mu}(\mathscr{B})] = 1 \quad \blacksquare$$

We now couple $(\tilde{\sigma}^{\tilde{\mu}}_t)_{t \geqslant 0}$ and $(\tilde{\sigma}^{-\underline{1}}_t)_{t \geqslant 0}$ in the following manner: the initial configuration of the former is chosen with respect to $\tilde{\mu}$; they evolve independently until they meet and after that they evolve together. We call it coupling B.

**Lemma 3.3.2.** We have $\lim_{\beta \to \infty} \tilde{\mu}(-\underline{1}) = 1$.

*Proof.* Consider $\Theta = \inf\{t \geqslant 0;\ \tilde{\sigma}^{\tilde{\mu}}_t = -\underline{1}\}$. By Lemma 3.3.1 we know that

$$\lim_{\beta \to \infty} \mathbb{P}(\Theta < e^{\beta(h+\delta_1)}) = 1, \qquad \forall \delta_1 > 0$$

On the other hand,

$$\lim_{\beta \to \infty} \mathbb{P}(\tilde{\sigma}^{-\underline{1}}_t \neq -\underline{1}, \text{ for some } t \in [0, e^{\beta(4-h-\delta_2)}]) = 0 \tag{3.3.6}$$

In this way,

$$\lim_{\beta \to \infty} \mathbb{P}(\tilde{\sigma}^{\tilde{\mu}}_t = \tilde{\sigma}^{-\underline{1}}_t, \text{ for some } t < e^{\beta(h+\delta_1)}) = 1$$

For $t \in [\exp\{\beta(h+\delta_1)\}, \exp\{\beta(4-h-\delta_2)\}]$ and coupling B we have

$$\begin{aligned}
&\lim_{\beta \to \infty} \mathbb{P}(\tilde{\sigma}^{-\underline{1}}_t \neq -\underline{1}) \\
&\quad = \lim_{\beta \to \infty} \{\mathbb{P}(\tilde{\sigma}^{-\underline{1}}_t \neq 1, \tilde{\sigma}^{-\underline{1}}_t = \tilde{\sigma}^{\tilde{\mu}}_t) + \mathbb{P}(\tilde{\sigma}^{-\underline{1}}_t \neq -\underline{1}, \tilde{\sigma}^{-\underline{1}}_t \neq \tilde{\sigma}^{\tilde{\mu}}_t)\} \\
&\quad \geqslant \lim_{\beta \to \infty} \mathbb{P}(\tilde{\sigma}^{-\underline{1}}_t \neq -\underline{1}, \tilde{\sigma}^{-\underline{1}}_t = \tilde{\sigma}^{\tilde{\mu}}_t) \\
&\quad = \lim_{\beta \to \infty} \mathbb{P}(\tilde{\sigma}^{\tilde{\mu}}_t \neq -\underline{1}) \\
&\quad = \tilde{\mu}(\mathscr{D} \setminus \{-\underline{1}\}) \geqslant 0
\end{aligned} \tag{3.3.7}$$

By (3.3.6) and (3.3.7) we have

$$\lim_{\beta \to \infty} \tilde{\mu}(-\underline{1}) = 1 \quad \blacksquare$$

We now prove (3.3.1) by showing the corresponding asymptotic loss of memory.

This consists in verifying that

$$\lim_{\beta \to \infty} |\Delta_\beta(s, t)| = 0 \tag{3.3.8}$$

where

$$\Delta_\beta(s, t) = \mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > (s+t)\,\gamma_\beta) - \mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > s\gamma_\beta)\,\mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > t\gamma_\beta)$$

As in ref. 6, using the Markov property, we easily get

$$\begin{aligned}
&\mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > (s+t)\,\gamma_\beta)\\
&\quad\leqslant \mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > s\gamma_\beta)\,\mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > t\gamma_\beta)\\
&\qquad + \sum_{\eta \in \mathscr{D}/\{-\underline{1}\}} \mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > s\gamma_\beta,\, \sigma^{-\underline{1}}_{s\gamma_\beta} = \eta)\,\mathbb{P}(T^{\eta}(\mathscr{C}) > t\gamma_\beta)
\end{aligned}$$

In this way we obtain

$$\begin{aligned}
\Delta_\beta(s, t) &\leqslant \mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > s\gamma_\beta,\, \sigma^{-\underline{1}}_{s\gamma_\beta} \neq -\underline{1})\\
&\leqslant \mathbb{P}(\tilde{\sigma}^{-\underline{1}}_{s\gamma_\beta} \neq -\underline{1})
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > (s+t)\,\gamma_\beta)\\
&\quad\geqslant \mathbb{P}(\sigma^{-\underline{1}}_{s\gamma_\beta} = -\underline{1},\, \sigma_0^{-\underline{1}} = \sigma_0^{\tilde{\mu}}/T^{-\underline{1}}(\mathscr{C}) > s\gamma_\beta)\\
&\qquad \times \mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > s\gamma_\beta)\,\mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > t\gamma_\beta)\\
&\qquad + \mathbb{P}(\sigma^{-\underline{1}}_{s\gamma_\beta} = -\underline{1},\, \sigma_0^{-\underline{1}} \neq \sigma_0^{\tilde{\mu}}/T^{-\underline{1}}(\mathscr{C}) > s\gamma_\beta)\\
&\qquad \times \mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > s\gamma_\beta)\,\mathbb{P}(T^{-\underline{1}}(\mathscr{C}) > t\gamma_\beta)
\end{aligned}$$

Now by coupling B,

$$\begin{aligned}
\mathbb{P}(\tilde{\sigma}^{-\underline{1}}_{s\gamma_\beta} \neq -\underline{1}) &= \mathbb{P}(\tilde{\sigma}^{-\underline{1}}_{s\gamma_\beta} \neq -\underline{1},\, \tilde{\sigma}_0^{-\underline{1}} = \tilde{\sigma}_0^{\tilde{\mu}}) + \mathbb{P}(\tilde{\sigma}^{-\underline{1}}_{s\gamma_\beta} \neq -\underline{1},\, \tilde{\sigma}_0^{-\underline{1}} \neq \tilde{\sigma}_0^{\tilde{\mu}})\\
&\leqslant \mathbb{P}(\tilde{\sigma}^{\tilde{\mu}}_{s\gamma_\beta} \neq -\underline{1}) + \mathbb{P}(\tilde{\sigma}_0^{-\underline{1}} \neq \tilde{\sigma}_0^{\tilde{\mu}})\\
&= \tilde{\mu}(\mathscr{D}/\{-\underline{1}\}) + \tilde{\mu}(\mathscr{D}/\{-\underline{1}\})
\end{aligned} \tag{3.3.9}$$

The above probability goes to zero when $\beta$ goes to infinity by Lemma 3.3.2 and we finish the proof.

To prove (3.3.3) we observe that since $T$ is larger than the time needed for the first spin to flip starting from $-\underline{1}$, we have

$$\gamma_\beta \geqslant e^{\beta(4-h)}$$

Now (3.3.3) follows using part (b) of Proposition 1.1. To replace $\gamma_\beta$ by $ET$, observe that, by monotonicity,

$$\mathbb{P}(T > \gamma_\beta u) \leqslant [\mathbb{P}(T > \gamma_\beta)]^u$$

Using (3.3.4), it follows that $\mathbb{P}(T > \gamma_\beta) < 1$ if $\beta$ is larger, so we can use dominated convergence below:

$$\lim_{\beta \to \infty} \frac{ET}{\gamma_\beta} = \lim_{\beta \to \infty} \frac{\int_0^\infty \mathbb{P}(T > t)\, dt}{\gamma_\beta} = \lim_{\beta \to \infty} \int_0^\infty \mathbb{P}\left(\frac{T}{\gamma_\beta} > u\right) du = 1 \quad \blacksquare$$

### 3.4. Proof of Theorem 1.3

As mentioned in the introduction, to get the stability of suitable time averages we need first a lower bound on the tunneling time $T^{-\underline{1}}(+\underline{1})$. As for the asymptotics of such a tunneling time, an important result can be obtained by ref. 8 and we recall it now.

**Theorem 3.4.1.** We have the following results.

(a) $\lim_{\beta \to \infty}(1/\beta) \log T^{-\underline{1}}(+\underline{1}) = 6 - h$ in probability.

(b) $\lim_{\beta \to \infty}(1/\beta) \log E(T^{-\underline{1}}(+\underline{1})) = 6 - h$.

*Proof.* The proof follows from the renormalization scheme developed in ref. 8.

*Basic Hypothesis.* Suppose we have a Markov chain with finite state space $\mathscr{S}$ and with transition probabilities satisfying the following condition: for any $\eta, \zeta \in \mathscr{S}$, with $\eta \neq \zeta$, if $P(\eta, \zeta) > 0$, then

$$\exp[-\Delta(\eta, \zeta)\, \bar{\beta} - \gamma\bar{\beta}] \leqslant P(\eta, \zeta) \leqslant \exp[-\Delta(\eta, \zeta)\, \bar{\beta} + \gamma\bar{\beta}]$$

where $\Delta(\eta, \zeta)$ assumes the values $\Delta_0 = 0 < \Delta_1 < \cdots < \Delta_n$, and $\gamma = \gamma(\bar{\beta}) \to 0$ as $\bar{\beta} \to \infty$.

*Iteration Scheme* (ref. 8, p. 101). We have ($\mathscr{R}^* = \{\eta \in X_N\colon\ \eta$ is stable$\}$)

$$\mathscr{S}^{(0)} = \{-1, +1\}^{\Lambda_N}, \qquad M^{(0)} = \{-\underline{1}, +\underline{1}, \mathscr{R}^*\}$$

$$\mathscr{S}^{(1)} = M^{(0)} = \{-\underline{1}, +\underline{1}, \mathscr{R}^*\}, \qquad M^{(1)} = \{-\underline{1}, +\underline{1}\}$$

$$\mathscr{S}^{(2)} = M^{(1)} = \{-\underline{1}, +\underline{1}\}, \qquad M^{(2)} = \{+\underline{1}\}$$

Consider $\phi$: $\mathbb{N} \to X_N$. We have

$$V_1 = \inf_{\substack{\eta \in M, \\ \zeta \in \mathcal{S}}} V(\eta, \zeta) = \inf_{\substack{\eta \in M, \\ \zeta \in \mathcal{S}}} \inf_{\substack{\phi, t \\ \phi_0 = \eta \\ \phi_t = \zeta}} I_{[0,t]}(\phi)$$

$$= \inf_{\substack{\eta \in M, \\ \zeta \in \mathcal{S}}} \inf_{\substack{\phi, t \\ \phi_0 = \eta \\ \phi_t = \zeta}} \sum_{i=0}^{t-1} \Delta(\phi_i, \phi_{i+1})$$

$$= h$$

$$V_2 = \inf_{\substack{\eta \in M^{(1)}, \\ \zeta \in \mathcal{S}^{(1)}}} V^{(1)}(\eta, \zeta)$$

$$= \inf_{\substack{\eta \in M^{(1)}, \\ \zeta \in \mathcal{S}^{(1)}}} \inf_{\substack{\phi, t, \\ \phi_0 = \eta, \\ \phi_t = \zeta}} I^{(1)}(\phi)$$

$$= \inf_{\substack{\eta \in M^1, \\ \zeta \in \mathcal{S}^1}} \inf_{\substack{\phi, t \\ \phi_0 = \eta, \\ \phi_t = \zeta}} \sum_{i=0}^{t-1} \bar{\Delta}^{(1)}(\phi_i, \phi_{i+1})$$

$$= \inf_{\substack{\eta \in M^{(1)}, \\ \zeta \in \mathcal{S}^{(1)}}} \inf_{\substack{\phi, t, \\ \phi_0 = \eta, \\ \phi_t = \zeta}} \sum_{i=0}^{t-1} \bar{\Delta}^{(0)}(\phi_i, \phi_{i+1}) - V_1, \qquad \phi_i, \phi_{i+1} \in \mathcal{S}^{(1)}$$

$$= 6 - 2h$$

Then from Theorem 2.1 in ref. 8 we have the proof of Theorem 3.4.1. ▌

**Lemma 3.4.1.** Let us call $I(\cdot)$ the indicator function of $-\underline{1}$ and take $R'_\beta = \exp[\beta(3-h)]$, $0 < h < 1$. Then for $\delta > 0$, there exists $\beta_0(\delta) < \infty$ such that

$$\sup_{\eta \in \mathcal{A} \cup \mathcal{B}} \mathbb{P}\left\{ \left| \frac{1}{R} \int_0^R I(\sigma_s^\eta)\, ds - 1 \right| > \delta,\ T^\eta(\mathscr{C}) > R \right\} \leqslant \exp\left( -c_\delta \frac{R}{R'_\beta} \right) \tag{3.4.1}$$

for $R > 2R'_\beta$ and $\beta > \beta_0$, where $c_\delta$ is a positive constant.

*Proof.* The proof follows from ref. 2. In fact, we use Proposition 1.1 to prove that

$$p(\beta) = \sup_{\eta \in \mathcal{A} \cup \mathcal{B}} \mathbb{P}(Y_\beta^1 = 1,\ T^\eta(\mathscr{C}) > t_\beta)$$

goes to zero when $\beta$ goes to infinity, where

$$Y^i_\beta = 0 \quad \text{if } \sigma_. \text{ visits } -\underline{1} \text{ during } ((i-1)\,t_\beta, (i-1)\,t_\beta + \sqrt{t_\beta}] \text{ and spends the rest of time interval } ((i-1)\,t_\beta, it_\beta] \text{ in } (-\underline{1})$$
$$= 1 \quad \text{otherwise}$$

for $i = 1,..., N_\beta$ and $t_\beta = \exp\{\beta(2h)\}$. ∎

**Lemma 3.4.2.** There exists $R_\beta \to \infty$ with $R_\beta/\gamma_\beta \to 0$ such that for all $\delta > 0$,

$$\sup_{\beta \in \mathscr{A}} \mathbb{P}\left\{ \sup_{0 < s < T^\eta(\mathscr{C}) - 2R_\beta} \left| \frac{1}{R_\beta} \int_s^{s+R_\beta} I(\sigma^\eta_u)\, du - 1 \right| > \delta \right\} \to 0 \qquad (3.4.2)$$

as $\beta \to \infty$.

*Proof.* The proof follows from ref. 2, with $R_\beta = \exp\{\beta(4-h)\}$. ∎

The above choice of $R_\beta$ also works if we reverse the roles of $-\underline{1}$ and $+\underline{1}$ in Lemma 3.4.2. This fact and the use of the strong Markov property at $\tilde{T}_\beta$ allows us to conclude the validity of (1.16). ∎

### 3.5. The Pattern of Escape

The estimate of $T^{-1}(+\underline{1})$ suggests how the process escapes from $-\underline{1}$. First note that the configuration in $\mathscr{B}$ has at least two positive spins and the configuration

$$\begin{array}{ccccccccc}
- & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - \\
- & - & - & - & + & - & - & - \\
- & - & - & + & - & - & - & - \\
- & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & -
\end{array}$$

$$\eta^*$$

belongs to $\mathscr{B}$.

In fact, any rotation or translation of $\eta^*$ belongs to $\mathscr{B}$. Denote this class by $\mathscr{M}^*$.

Now consider $\bar{\mathscr{M}}^2$, the class of configurations with two positive spins, and $\mathscr{M}^2$, the class of configurations with two neighbor positive spins ($\mathscr{M}^2 \subset \bar{\mathscr{M}}^2$). We have the following result.

**Theorem 3.5.1.** We have

$$\lim_{\beta \to \infty} \mathbb{P}(\sigma_{T^{-1}(\bar{\mathcal{M}}^2)}^{-1} \in \mathcal{M}^2; \sigma_{T^{-1}(\mathcal{B})}^{-1} \in \mathcal{M}^*) = 1$$

*Proof.* The proof of this result is easy from Theorem 3.4.1.
Suppose that

$$\sigma_{T^{-1}(\bar{\mathcal{M}}^2)}^{-1} \notin \mathcal{M}^2$$

Until $T^{-1}(\bar{\mathcal{M}}^2)$ the exclusion process only translates the configurations with one positive spin, so that from the results for the Glauber dynamics[6] we have that the necessary time for the creation of two non-neighbor positive spins is larger than $\exp\{\beta(8-2h-\varepsilon)\}$ with probability close to one, for any positive $\varepsilon$, when $\beta$ goes to infinity.

Now, suppose that

$$\sigma_{T^{-1}(\mathcal{B})}^{-1} \notin \mathcal{M}^*$$

In this manner, this configuration has at least three positive spins. But this creation, without reaching $\mathcal{M}^*$ (i.e., the stirring process does not occur in the configuration with two positive spins at neighbor sites), needs a time larger than $\exp\{\beta(8-3h-\delta)\}$, with probability tending to one, for any positive $\delta$, when $\beta$ tends to infinity, because only the Glauber dynamics creates positive spins.

Since $8-2h>6-h$ and $8-3h>6-h$ for any $h<1$, we have the result. ∎

Notice that Theorem 3.5.1 implies that the escape from $-\underline{1}$ to $\mathcal{B}$ does not follow the reversed path, in contrast with the Glauber dynamics.

## 4. OTHER EXCLUSION RATES

Now we to consider the exclusion rate $c(x, y, \eta) = e^{-\beta(2-h-\delta)}$, where $h < 2-h-\delta \leqslant 2-h$ (that is, in this case the stirring mechanism does not compete with the erosion of corners on Glauber dynamics).

In these cases, the critical droplet has a size that depends on the magnetic field $h$. The model, for each $\delta > 0$, has the metastable behavior.

### Results

**Theorem 4.1.** Suppose that $0<h<1$ and $\eta \in \mathcal{R}$.

(a) If $l(\eta) < (2-\delta)/h$, then for $\varepsilon > 0$,

$$\lim_{\beta \to \infty} \mathbb{P}(T^{\eta}(-\underline{1}) < T^{\eta}(+\underline{1}), T^{\eta}(-\underline{1}) \leqslant e^{\beta(h(l(\eta)-1)+\varepsilon)}) = 1$$

(b) If $l(\eta) > (2-\delta)/h$ and $l(\eta) > 3$, then for $\varepsilon > 0$,

$$\lim_{\beta \to \infty} \mathbb{P}(T^\eta(+\underline{1}) < T^\eta(-\underline{1}), T^\eta(+\underline{1}) \leqslant e^{\beta(2-h-\delta+\varepsilon)}) = 1$$

(c) If $l(\eta) > (2-\delta)/h$ and $l(\eta) \leqslant 3$, then for $\varepsilon > 0$,

$$\lim_{\beta \to \infty} \mathbb{P}(T^\eta(-\underline{1}) < T^\eta(+\underline{1})) > 0$$

$$\lim_{\beta \to \infty} \mathbb{P}(T^\eta(+\underline{1}) < T^\eta(-\underline{1})) > 0$$

$$\lim_{\beta \to \infty} \mathbb{P}(T^\eta\{-\underline{1}, +\underline{1}\} \leqslant e^{\beta(2-h-\delta+\varepsilon)}) = 1$$

Note that if $\delta = 0$, i.e., $c(x, y, \eta) = e^{-\beta(2-h)}$, we have that the droplet has a very precise shape and size. In this sense we have a continuity in $\delta$ of the size of the droplet. In fact, if $l \geqslant 4$, the set $\mathscr{B}$ is similar to case studied in ref. 6.

Theorem 4.1 has as a consequence the possibility of dividing the set of all configurations into three nonempty sets $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, as in Proposition 1.1 and the analog of Theorem 1.3.

**Theorem 4.2.** Consider $T = \inf\{t \geqslant 0 : \sigma_t^{-\underline{1}} = +\underline{1}\}$; then,

$$\frac{T}{ET} \to \tau \quad \text{in distribution as} \quad \beta \to \infty$$

where $\tau$ is a unit mean exponential random variable.

Theorem 4.2 characterizes the metastable behavior of the new models.

To prove Theorems 4.1 and 4.2, it is enough to modify a few things in the proofs of Theorems 1.1 and 1.2. In these cases the equivalence classes in the renormalization procedure of ref. 8 are unitary sets and so, for $(2-\delta)/h > 3$, the results also follow from ref. 8.

## ACKNOWLEDGMENTS

Special thanks are due to M. E. Vares for her accurate and constructive criticism. I also thank E. J. Neves for many important discussions and R. H. Schonmann for many suggestions. This work is part of the author's Ph.D. thesis. This research has been partially supported by FAPESP 92/1212-3, IME-USP, and IMPA (RJ).

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